

# The transformations of spinors<sup>1</sup>

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**Abstract.** We begin showing that for even dimensional vector spaces  $V$  all automorphisms of their Clifford algebras are inner. So all orthogonal transformations of  $V$  are restrictions to  $V$  of inner automorphisms of the algebra. Thus under orthogonal transformations  $P$  and  $T$  — space and time reversal — all algebra elements, including vectors  $v$  and spinors  $\varphi$ , transform as  $v \rightarrow xvx^{-1}$  and  $\varphi \rightarrow x\varphi x^{-1}$  for some algebra element  $x$ . We show that while under combined  $PT$  spinor  $\varphi \rightarrow x\varphi x^{-1}$  remain in its spinor space, under  $P$  or  $T$  separately  $\varphi$  goes to a *different* spinor space and may have opposite chirality. We conclude with a preliminary characterization of inner automorphisms with respect to their property to change, or not, spinor spaces.

## 1. Introduction

Élie Cartan introduced spinors in 1913 [7, 8] and, after more than a century, this field still yields good harvests. Spinors were later thoroughly investigated by Claude Chevalley [10] in the mathematical frame of Clifford algebras where they were identified as elements of Minimal Left Ideals (MLI) of the algebra. Many years later Benn and Tucker [1] and Porteous [14] wrote books with many of these results easier to be assimilated by physicists.

In this paper we address the transformation properties of spinors under certain inner automorphisms of Clifford algebra exploiting the Extended Fock Basis (EFB) of Clifford algebra [2, 3, 4]. With this formalism one can write vectors as linear superpositions of simple spinors, thus supporting the well-known Penrose twistor program [13] that spinor structure is the underlying — more fundamental — structure of Minkowski spacetime.

In section 2 we review some quite general properties of a simple Clifford algebra and in particular the fact that it contains many different MLI, namely many different spinor spaces, that are completely equivalent in the sense that each of them can be the carrier of a representation; moreover the algebra, as a vector space, is the direct sum of these spinor spaces. These properties are known and recently Pavsic suggested that multiple spinor spaces play a role in physics [12].

One of the most important properties of Clifford algebra is that it establishes a deep connection between the orthogonal transformations of vector space  $V$  with scalar product  $g$  (more precisely: its image in the algebra) and the automorphisms of Clifford algebra  $\mathcal{Cl}(g)$ . In section 3 we show first that if the vector space is even dimensional then all  $\mathcal{Cl}(g)$  automorphisms are *inner* automorphisms and thus that all orthogonal transformations on  $V$  lift to inner automorphisms of  $\mathcal{Cl}(g)$ . We then examine in detail the so called discrete orthogonal transformations of  $V$ , namely  $1_V, P, T$  and  $PT$  ( $V$  identity, space and time reversal

<sup>1</sup> an older version of this paper [4] had been submitted before IARD 2016 and was accepted after the conference. This version is newer with some parts removed and with some new results.



and their composition) and we focus on the inner algebra automorphism they induce. This study takes advantage of the properties of the EFB that allow to remain within the algebra without using representations. At the same time we exhibit the elements of the algebra that generate these inner automorphisms. It follows that we can look at  $\mathbb{1}_V, P, T$  and  $PT$  as at restrictions of automorphisms of the entire algebra to  $V$ , thus unifying the treatment of the discrete transformations of  $V$  with those of the continuous ones of the  $\text{Pin}(g)$  group.

The same approach was taken also by Varlamov [16, 17] to study the hierarchies of  $\text{Pin}(g)$  and  $\text{O}(g)$  groups and he successfully classified the automorphisms of  $\mathcal{Cl}(g)$  showing that the eight double coverings of  $\text{O}(g)$ , the Dabrowski groups [11], correspond to the eight types of real Clifford algebras: the so called “spinorial clock” [5].

Here we exploit the same unification to investigate a different subject: given an inner automorphism

$$\alpha : \mathcal{Cl}(g) \rightarrow \mathcal{Cl}(g); \quad \alpha(\mu) = x\mu x^{-1} \quad x \in \mathcal{Cl}(g)$$

it is manifest that all algebra elements must transform accordingly and in particular that the typical physics equations  $v\varphi = 0$ , where  $v \in V$  and  $\varphi$  is a spinor, must go to  $\alpha(v\varphi) = 0$ . We remark that  $\varphi$  is *both* a carrier of the regular representation *and* an element of  $\mathcal{Cl}(g)$  so the equation  $\alpha(v\varphi) = 0$  is justified. Since the automorphism is inner it follows that both  $v$  and spinor  $\varphi$  must transform as  $\alpha(\varphi) = x\varphi x^{-1}$  thus adding an “extra”  $x^{-1}$  to the “traditional rule” stating that vectors transform as  $v \rightarrow vxv^{-1}$  while spinors as  $\varphi \rightarrow x\varphi$ . This consequence is unavoidable if we accept that spinors are part of the Clifford algebra and not elements of some “external” linear space, a point of view that, even if historical, is rarely taken nowadays.

We examine in detail the spinors transformations  $\alpha(\varphi) = x\varphi x^{-1}$  proving that if on one side they can not alter in any way the solutions of  $v\varphi = 0$ , on the other hand, in some cases, they “move”  $\varphi$  to a different spinor space, one of the many equivalent ones in  $\mathcal{Cl}(g)$ . In particular we show that while the automorphisms corresponding to  $\mathbb{1}_V$  and  $PT$  do not move spinors, those corresponding to  $P$  and  $T$  move them, thus populating other spinor spaces.

In section 4 we begin the characterization of these automorphisms: those that keep the spinor space constant, like  $\mathbb{1}_V$  and  $PT$  and those that do not, like  $P$  and  $T$ , and we show that the latter transformations can also invert spinor chiralities. This is the first part of the study of these transformations that will be completed examining also continuous transformations.

## 2. Multiple spinor spaces

We consider Clifford algebras  $\mathcal{Cl}(g)$  [10, 14, 5] over the fields  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$  and an *even* dimensional vector space  $V$  equipped with a non degenerate scalar product  $g$ ; any base  $e_1, e_2, \dots, e_n$  with  $n = 2m$  generates the algebra that results: simple, central and of dimension  $2^n$ .

EFB formalism is fully developed for *neutral* spaces:  $V = \mathbb{C}^{2m}$  or  $\mathbb{R}^{m,m}$ , spaces for which Witt decomposition is the direct sum of two totally null (isotropic) subspaces of dimension  $m$ ; when we refer to this case we indicate the corresponding Clifford algebra  $\mathcal{Cl}(m, m)$ . This choice allows to treat a simpler case, avoiding the many intricacies brought in by other signatures (the extension of the formalism to other cases is under development). At the same time this restriction is much milder than it may seem since the following results apply also to the complexification of the Clifford algebras of even dimensional real spaces of *any* signature.

In neutral spaces the  $e_i$ ’s form an orthonormal basis of  $V$  with e.g.

$$2g(e_i, e_j) = e_i e_j + e_j e_i := \{e_i, e_j\} := g_{ij} = 2\delta_{ij}(-1)^{i+1}$$

while  $\{e^i, e_j\} = 2\delta_j^i$  and

$$\begin{cases} e_{2i-1}^2 &= 1 \\ e_{2i}^2 &= -1 \end{cases} \quad i = 1, \dots, m \quad . \quad (1)$$

The Witt, or null, basis of the vector space  $V$  is defined, for both fields:

$$\begin{cases} p_i &= \frac{1}{2}(e_{2i-1} + e_{2i}) \\ q_i &= \frac{1}{2}(e_{2i-1} - e_{2i}) \end{cases} \Rightarrow \begin{cases} e_{2i-1} &= p_i + q_i \\ e_{2i} &= p_i - q_i \end{cases} \quad i = 1, 2, \dots, m \quad . \quad (2)$$

The EFB of  $\mathcal{Cl}(m, m)$  is given by the  $2^{2m}$  different sequences

$$\psi_1 \psi_2 \cdots \psi_m := \Psi \quad \psi_i \in \{q_i p_i, p_i q_i, p_i, q_i\} \quad i = 1, \dots, m \quad (3)$$

in which each  $\psi_i$  can take four different values and we reserve  $\Psi$  for EFB elements and  $\psi_i$  for its components. The main characteristics of EFB is that all its  $2^{2m}$  elements  $\Psi$  are simple spinors [6]. Each element of this basis of the algebra is completely determined by two binary signatures,  $h$  and  $h \circ g$  signatures,  $a$  and  $b$ , [4], that represent also row and column indexes of the matrix algebra; we will thus indicate one element of this basis with  $\Psi_{ab}$ .

$\mathcal{Cl}(m, m)$ , as a vector space, is the direct sum of subspaces of different  $h \circ g$  signatures [3]. Given the Clifford product properties in EFB formalism [4] these subspaces are also MLI of  $\mathcal{Cl}(m, m)$  and thus coincide with  $2^m$  different spinor spaces  $S_{hg}$  that in turn correspond to different columns of the isomorphic matrix algebra  $\mathbb{F}(2^m)$ . To establish a further link between EFB and the familiar definition of MLI in  $\mathcal{Cl}(m, m)$  [1] we remark that in EFB the  $2^m$  elements with identical  $h$  and  $h \circ g$  signatures [4], namely  $\Psi_{aa}$ , are primitive idempotents and the MLI  $S_a$  can thus be written as  $S_a = \mathcal{Cl}(m, m) \Psi_{aa}$ .

Each of the  $2^m$  spinor spaces supports a regular, faithful and irreducible representation of  $\mathcal{Cl}(m, m)$  and since the algebra is simple there exist isomorphisms intertwining the representations. This has been known for a long time but recently mirror particles [12] have been proposed as a possible realization of multiple spinor spaces. Here we show how, under certain transformations, e.g.  $P$  and  $T$ , a spinor moves to a different spinor space.

We choose a particular spinor space, e.g.  $h \circ g = f$  so that when speaking of a generic spinor we will refer to spinor space  $S_f$  (used to build the Fock basis in [6]). Its generic element  $\varphi \in S_f$  can thus be expanded in the Fock basis

$$\varphi = \sum_a \xi_{af} \Psi_{af} \quad (4)$$

and, since the second index  $f$  is constant, in principle it could be omitted.

Let this  $\varphi \in S_f$  be a solution of the Weyl equation  $v\varphi = 0$  where  $v \in V$ ; we remark that the equation is solved also by all  $\varphi' = \sum_a \xi_a \Psi_{af'} \in S_{f'}$  for any  $f'$ ; we will return to this point in paragraph 4.

### 2.1. Representations of Clifford algebra $\mathcal{Cl}(g)$

We resume some quite general properties we need in the sequel: let  $\gamma : \mathcal{Cl}(g) \rightarrow \text{End} S$  be a faithful irreducible representation of  $\mathcal{Cl}(g)$  and let  $\beta$  be the so called main antiautomorphism

$$\begin{cases} \beta(\mu\nu) &= \nu\mu & \forall \mu, \nu \in \mathcal{Cl}(g) \\ \beta(v) &= v & \forall v \in V \\ \beta(1) &= 1 \end{cases} \quad (5)$$

that reverses the order of multiplication and that is involutive. With  $\beta$  it is possible to define the contragredient representation in  $S^*$ , the dual of  $S$ ,  $\check{\gamma} : \mathcal{Cl}(g) \rightarrow \text{End} S^*$  given by  $\check{\gamma}(\mu) = \gamma(\beta(\mu))^*$  and, since in our case  $V$  is even dimensional,  $\mathcal{Cl}(g)$  is simple and central and thus there exists an isomorphism  $B : S \rightarrow S^*$  intertwining the two representations:  $\check{\gamma}B = B\gamma$  which is either

symmetric or antisymmetric  $B = \pm B^*$  [5, 15, 9] and that also defines on  $S$  the structure of an inner product space ( $\langle \cdot, \cdot \rangle$  represents the bilinear product or contraction)

$$S \times S \rightarrow \mathbb{F} \quad B(\varphi, \phi) := \langle B\varphi, \phi \rangle \in \mathbb{F} .$$

This structure extends to  $\text{End}S$ : there is a symmetric isomorphism  $B \otimes B^{-1} : \text{End}S \rightarrow (\text{End}S)^* = \text{End}S^*$  given, for every  $\gamma \in \text{End}S$ , by  $(B \otimes B^{-1})(\gamma) = B\gamma B^{-1}$ . These results are fully general and hold thus also when  $\gamma$  is the regular representation  $\gamma(\mu) = \mu$ ,  $\text{End}S = \mathcal{C}\ell(g)$  and  $S$  is one of its MLI.

### 3. Automorphisms of Clifford algebra $\mathcal{C}\ell(g)$

We begin with a general proposition and thus in this section there are no restrictions on the dimensions of the vector space  $V$ .

**Proposition 1.** *For a Clifford algebra over fields  $\mathbb{R}$  and  $\mathbb{C}$  all its automorphisms are inner if and only if the dimension of the vector space is even.*

*Proof.* For any non degenerate, even dimensional, vector space  $V$  its Clifford algebra is central and simple [5] and, by Skolem – Noether theorem, all its automorphisms are inner. To prove the converse we take an odd dimensional vector space and we examine the so called main automorphism of its Clifford algebra, the automorphism that reverses all vectors (7). In this case the volume element  $\omega = e_1 e_2 \cdots e_n$  is formed by an odd number of vectors and thus the main automorphism sends  $\omega \rightarrow -\omega$ . But in this case  $\omega$  belongs also to the center of the algebra and thus for any inner automorphism  $x\omega x^{-1} = \omega$ , thus the main automorphism is not inner.  $\square$

A simple example is  $\mathcal{C}\ell_{\mathbb{R}}(0, 1) \cong \mathbb{C}$  where the main automorphism coincides with complex conjugation and is not inner. In general the Clifford algebra of an odd dimensional vector space has the form  $\mathcal{C}\ell(g) \oplus \mathcal{C}\ell(g)$  and have no inner elements giving the main automorphism, nevertheless the main automorphism is obtained by the “swap” automorphism  $\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$  that is not an element of the algebra but is an element of the matrix representations of the algebra.

A corollary that follows from the universality of Clifford algebra is that all orthogonal transformations on an even dimensional  $V$  lift to inner automorphisms of  $\mathcal{C}\ell(g)$ . This corollary gives a simpler proof, but only for even dimensional spaces, of the result quoted in [16] that all “fundamental automorphisms”, even discrete ones like  $P$  and  $T$ , are inner automorphisms.

So in even dimensional spaces

$$\text{Aut}(\mathcal{C}\ell(g)) = \{x \in \mathcal{C}\ell(g) : \exists x^{-1}\} := C_g^* \quad (6)$$

and the Clifford Lipschitz group is its subgroup that stabilizes vectors that in turn, when restricted on vector space, is the orthogonal group  $O(g)$ .

#### 3.1. Fundamental automorphisms of $\mathcal{C}\ell(g)$

In general in  $\mathcal{C}\ell(g)$  there are four automorphisms corresponding to the two involutions and to the two antinvolutions induced by the orthogonal transformations  $\mathbb{1}_V$  and  $-\mathbb{1}_V$  of vector space  $V$  [14, Theorem 15.32]. They are called fundamental or discrete automorphisms and form a finite group, isomorphic to the Klein four group  $\mathbb{Z}_2 \otimes \mathbb{Z}_2$  [16]. We review them briefly to show how they appear in EFB formalism and to exhibit the elements of  $C_g^*$  realizing the inner automorphisms in even dimensional vector spaces.

From now on we restrict again to even dimensional, neutral, spaces to fully exploit EFB properties.

### 3.2. Identity automorphism of $\mathcal{Cl}(g)$

The just quoted theorem proves also that  $\mathbb{1}_V$  induces the algebra identity automorphism, its internal element in  $\mathcal{Cl}(m, m)$  is  $\mathbb{1}$ .

### 3.3. Main automorphism of $\mathcal{Cl}(g)$

The main automorphism  $\alpha$  of  $\mathcal{Cl}(g)$  (main involution in [14]) is induced by  $V$  orthogonal transformation  $-\mathbb{1}_V$ , namely

$$\alpha(v) = -v \quad \forall v \in V \quad (7)$$

it is involutive and defines the basic  $\mathbb{Z}_2$  grading of  $\mathcal{Cl}(g)$ .

It's easy to see that given the volume element  $\omega = e_1 \cdots e_n$  we obtain  $\omega e_i = (-1)^{(n-1)} e_i \omega$  and so, for even dimensional spaces, we have  $\omega v \omega^{-1} = -v$  for any  $v \in V$ , where  $\omega^{-1} = \omega^3 = \pm \omega$  and thus in this case the main automorphism on the entire algebra may be written as:

$$\alpha(\mu) = \omega \mu \omega^{-1} \quad .$$

For the EFB expansion  $\mu = \sum_{ab} \xi_{ab} \Psi_{ab}$  we find

$$\alpha(\Psi_{ab}) = \omega \Psi_{ab} \omega^{-1} = \theta_{ab} \Psi_{ab} = \epsilon(a) \epsilon(b) \Psi_{ab} \quad (8)$$

where  $\theta_{ab} = \pm 1$  is the global parity of the EFB element  $\Psi_{ab}$  that can be expressed, defining the function  $\epsilon : \{\pm 1\}^m \rightarrow \{\pm 1\}$ ,  $\epsilon(h) = \prod_{i=1}^m h_i$ , from the  $h$  and  $g$  signatures of EFB, namely  $a$  and  $b$  [4].

### 3.4. Reversion automorphism of $\mathcal{Cl}(g)$

The main antiautomorphism  $\beta$  (5) is the antiautomorphism induced by the orthogonal transformation  $\mathbb{1}_V$  of  $V$  (reversion antiautomorphism in [14]) and gives an automorphism when “transposed to the dual space”. If  $S_f$  is a MLI of  $\mathcal{Cl}(g)$ , the space of spinors, and  $\gamma$  the regular representation  $\gamma(\mu) = \mu$ , then  $\check{\gamma}(\mu) = (\beta(\mu))^*$  is the contragredient representation that defines also the reversion automorphism; with (5) we get its main property:

$$\check{\gamma}(e_{i_1} \cdots e_{i_k}) = e_{i_1}^* \cdots e_{i_k}^* \quad .$$

Since it is an automorphism it must be inner, thus there exists  $\tau \in \mathcal{Cl}(g)$  such that  $\check{\gamma}(\mu) = \tau \mu \tau^{-1}$  and  $\tau$  is fully defined by its action on the generators  $\check{\gamma}(e_i) = e_i^* = \tau e_i \tau^{-1}$  and since  $\{e_i^*, e_j\} = g_j^i = 2\delta_j^i$  it follows that

$$\tau e_i \tau^{-1} = e_i^* = e_i^{-1} = e_i^3 \quad .$$

With this result and remembering (1) and (2) it is simple to get the explicit form of  $\tau$  that depends on the parity of  $m$

$$\begin{aligned} \tau &= \begin{cases} e_2 e_4 \cdots e_{2m} & \text{for } m \text{ even} \\ e_1 e_3 \cdots e_{2m-1} & \text{for } m \text{ odd} \end{cases} \\ &= (p_1 + s q_1) (p_2 + s q_2) \cdots (p_m + s q_m) \quad s = (-1)^{m+1} \quad . \end{aligned}$$

To evaluate the reversion automorphism on EFB elements we easily get

$$\check{\gamma}(\Psi_{ab}) = \beta(\Psi_{ab})^* = \beta(\psi_1 \psi_2 \cdots \psi_m)^* = \beta(\psi_1)^* \beta(\psi_2)^* \cdots \beta(\psi_m)^* \quad . \quad (9)$$

By (5)  $\beta(p_i) = p_i$ ,  $\beta(q_i) = q_i$  so that  $\beta(p_i q_i) = q_i p_i$  and  $\beta(q_i p_i) = p_i q_i$ . Since  $e_i^* = e_i^{-1}$  by (2) we obtain that  $(p_i)^* = q_i$ ,  $(p_i q_i)^* = p_i q_i$  and  $(q_i p_i)^* = q_i p_i$ . We can resume saying that the  $g_i$  and

$h_i$  signatures of  $\beta(\psi_i)^*$  are respectively equal to  $g_i$  and  $-h_i$  of that of  $\psi_i$  so that the effect of reversion automorphism is to change sign to both  $h$  and  $h \circ g$  signatures. We can thus conclude that for the reversion automorphism we have

$$\beta(\Psi_{ab})^* = \tau \Psi_{ab} \tau^{-1} = \Psi_{-a-b} \quad (10)$$

and we remark that while  $\Psi_{ab}$  belongs to spinor space  $S_b$ ,  $\beta(\Psi_{ab})^*$  belongs to  $S_{-b}$ , always a *different* spinor space, the main result of this paper.

For completeness we report the results of similar exercises:

$$\beta(\Psi_{ab}) = s'(a, b) \Psi_{-b-a} \quad s'(a, b) = \pm 1 \quad (11)$$

where the sign  $s'(a, b)$ , straightforward, if slightly tedious to calculate, depends on the indices; it is easy to double check that it satisfies the properties of the main antiautomorphism (5). We also obtain

$$\Psi_{ab}^* = s'(a, b) \Psi_{ba} \quad s'(a, b) = \pm 1 \quad (12)$$

that could also be deduced directly from the natural matrix structure of the EFB since  $a$  and  $b$  are just the matrix indexes written in binary form; combining both these formulas we reobtain (10). Since both (11) and (12) are involutive we have

$$s'(a, b) = s'(b, a) = s'(-b, -a) = s'(-a, -b) .$$

### 3.5. Conjugation automorphism of $\mathcal{Cl}(g)$

The composition of the main (8) and reversion automorphisms (10) is called conjugation and results in

$$\alpha(\beta(\Psi_{ab})^*) = \omega \tau \Psi_{ab} (\omega \tau)^{-1} = \theta_{ab} \Psi_{-a-b} \quad (13)$$

since  $\theta_{-a-b} = \theta_{ab}$  given that also this automorphism is involutive; clearly  $\alpha(\beta(e_i)^*) = -e_i^{-1}$ .

### 3.6. A simple example in $\mathcal{Cl}(1, 1)$

We conclude with a simple example in  $\mathcal{Cl}(1, 1) \cong \mathbb{R}(2)$  where the EFB is formed by 4 elements:  $\{qp_{++}, pq_{--}, p_{-+}, q_{+-}\}$  with the subscripts indicating respectively  $h$  and  $h \circ g$  signatures; its EFB matrix is

$$\begin{array}{cc} + & - \\ + & \begin{pmatrix} qp & q \\ p & pq \end{pmatrix} \\ - & \end{array}$$

and the generic element  $\mu \in \mathcal{Cl}(1, 1)$  can be written in the EFB as

$$\mu = \xi_{++} qp_{++} + \xi_{--} pq_{--} + \xi_{-+} p_{-+} + \xi_{+-} q_{+-} \quad \xi \in \mathbb{F}$$

and the application of the three inner automorphisms gives

$$\begin{aligned} \omega \mu \omega^{-1} &= \xi_{++} qp_{++} + \xi_{--} pq_{--} - \xi_{-+} p_{-+} - \xi_{+-} q_{+-} \\ \tau \mu \tau^{-1} &= \xi_{--} qp_{++} + \xi_{++} pq_{--} + \xi_{-+} p_{-+} + \xi_{+-} q_{+-} \\ \omega \tau \mu (\omega \tau)^{-1} &= \xi_{--} qp_{++} + \xi_{++} pq_{--} - \xi_{-+} p_{-+} - \xi_{+-} q_{+-} \end{aligned}$$

and  $\omega = e_1 e_2 = [q, p]$ ,  $\tau = e_1 = p + q$ ,  $\omega \tau = -e_2 = q - p$  and  $\mathbb{1} = e_1^2 = \{q, p\}$ . For comparison, the same automorphisms applied to the standard  $e_i$  formulation gives the ordinary results

$$\begin{aligned} \mu &= \xi_0 \mathbb{1} + \xi_1 e_1 + \xi_2 e_2 + \xi_{12} e_1 e_2 & \xi \in \mathbb{F} \\ \omega \mu \omega^{-1} &= \xi_0 \mathbb{1} - \xi_1 e_1 - \xi_2 e_2 + \xi_{12} e_1 e_2 \\ \tau \mu \tau^{-1} &= \xi_0 \mathbb{1} + \xi_1 e_1 - \xi_2 e_2 - \xi_{12} e_1 e_2 \\ \omega \tau \mu (\omega \tau)^{-1} &= \xi_0 \mathbb{1} - \xi_1 e_1 + \xi_2 e_2 - \xi_{12} e_1 e_2 . \end{aligned}$$

#### 4. Spinors transformations

We begin by observing that in the complex case, and also in complex representations of the real case, also complex conjugation is responsible for an automorphism and things get more complicated: the finite group of fundamental automorphism doubles its size and has been examined in detail in [17]. We leave aside for the moment complex conjugation and examine what's going on in our simpler case since it is enough for our purpose of introducing general properties of spinor transformations.

The inner automorphisms of section 3 are fully general and their restrictions to  $V$  correspond to the  $V$  transformations:  $\mathbb{1}_V$ ,  $P$ ,  $T$  and  $PT$ . We already saw that the restriction of the algebra identity to  $V$  is  $\mathbb{1}_V$  while we identified the main automorphism (7), (8) with  $PT$ . If we accept that  $T$  changes sign to timelike  $e_{2i}$  than reversion (10) and conjugation (13) restricted to  $V$  correspond respectively to  $T$  and  $P$  but other identifications are possible. A word of caution on this point: when we deal with complex representation of real algebras, how it is customary to do for Dirac spinors, a Wick rotation can easily swap timelike and spacelike vectors, e.g. going from  $\mathbb{R}^{3,1}$  to  $\mathbb{R}^{1,3}$ . Things would be different for real Clifford algebras but in this case our formalism take us to consider neutral spaces,  $\mathbb{R}^{m,m}$  and again the identification of timelike and spacelike coordinates is ambiguous. On top of that there is the fact that the automorphism group of  $\{\mathbb{1}_V, P, T, PT\} \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$  is the group of permutations of  $\{P, T, PT\}$  that thus can be freely permuted, and so there are no indications neither from this side. Anyhow whatever the precise identification of  $P$  and  $T$  their corresponding automorphisms both move also the spinorial space supporting the regular representation of  $\mathcal{Cl}(g)$  both sending  $\Psi_{ab}$  to  $\Psi_{-a-b}$  and in any case  $b \neq -b$ .

It is evocative to write the general form of these elements in EFB

$$\begin{aligned}\mathbb{1} &= \{q_1, p_1\} \{q_2, p_2\} \cdots \{q_m, p_m\} \\ \omega &= [q_1, p_1] [q_2, p_2] \cdots [q_m, p_m] \\ \tau &= (p_1 + s q_1) (p_2 + s q_2) \cdots (p_m + s q_m) \quad s = (-1)^{m+1} \\ \omega\tau &= (-1)^m (p_1 - s q_1) (p_2 - s q_2) \cdots (p_m - s q_m)\end{aligned}$$

and the first two result: even under the main automorphism, they do not move spinor spaces and form a group isomorphic to  $\mathbb{Z}_2$  that provides  $\mathcal{Cl}(g)$  grading. The last two, have parity  $(-1)^m$  under the main automorphism, move spinor spaces and form, together with the first two, the group of discrete automorphisms isomorphic to  $\mathbb{Z}_2 \otimes \mathbb{Z}_2$  group. Moreover since  $\eta(\Psi_{ab}) = \epsilon(a)$  and  $\theta(\Psi_{ab}) = \epsilon(a)\epsilon(b)$  [4] and observing that  $\epsilon(-x) = (-1)^m \epsilon(x)$  we find

$$\begin{aligned}\eta(\Psi_{-a-b}) &= (-1)^m \eta(\Psi_{ab}) \\ \theta(\Psi_{-a-b}) &= \theta(\Psi_{ab})\end{aligned}$$

showing that, when  $m$  is odd, the chirality is reversed by automorphisms that move spinor spaces; a subtler study is needed to generalize this result from neutral spaces to real spaces of different signature.

To investigate how these inner automorphisms behave on generic spinors (4) and not only on EFB elements we give a simple result:

**Proposition 2.** *for any inner automorphism  $\alpha \in C_g^*$  the image of a MLI is a MLI, moreover  $xS_{hg}x^{-1} = S_{hg}$  if and only if  $S_{hg}x^{-1} = S_{hg}$ .*

*Proof.* The first part follows immediately from the fact that a MLI must have rank 1; for the second part let  $xS_{hg}x^{-1} = S_{hg}$ , then since  $S_{hg}$  is a MLI we have also  $S_{hg}x^{-1} = S_{hg}$ ; vice versa let  $S_{hg}x^{-1} = S_{hg}$ , since  $S_{hg}$  is a MLI  $xS_{hg} = S_{hg}$  and thus  $xS_{hg}x^{-1} = S_{hg}$ .  $\square$



Thus since for any spinor  $\varphi = \varphi \mathbb{1}$  the identity do not change spinor space. Going to  $PT$  since  $\Psi_{af} = \epsilon(f)\Psi_{af}$  [4] and remembering that  $\omega^{-1} = \omega^3$

$$\varphi\omega^{-1} = \omega^2 \sum_a \xi_{af} \Psi_{af} \omega = \omega^2 \sum_a \xi_{af} \epsilon(f) \Psi_{af} = \omega^2 \epsilon(f) \varphi = \pm \varphi$$

and so both  $1_V$  and  $PT$  behave as expected also on generic spinors of any  $S_f$ . The effect of reversion automorphism (10) on a generic spinor is

$$\tau\varphi\tau^{-1} = \sum_a \xi_{af} \Psi_{-a-f}$$

and if  $\varphi$  has a defined chirality,  $\omega\varphi = \eta\varphi$ , then

$$\omega\tau\varphi\tau^{-1} = \sum_a \epsilon(-a) \xi_{af} \Psi_{-a-f} = (-1)^m \eta\tau\varphi\tau^{-1}.$$

We consider now the solutions of equations like  $v\varphi = 0$ , where  $v \in V$  and  $\varphi \in S$ . We observe that they must remain the same under any injective map and thus  $xv\varphi x^{-1} = 0$  if and only if  $v\varphi = 0$  and thus  $\varphi x^{-1} = 0$  only if  $\varphi = 0$ . As it was intuitive, inner automorphisms do not change solutions of  $v\varphi = 0$ ; in particular the solutions of  $xv\varphi = 0$  are identical to those of  $xv\varphi x^{-1} = 0$ .

We conclude with a first characterization of transformations that stabilize spinor spaces:

**Proposition 3.** *The automorphisms such that  $xS_fx^{-1} = S_f$  form a subgroup of  $C_g^*$  (6).*

*Proof.* Let  $x \in C_f := \{x \in C_g^* : xS_fx^{-1} = S_f\}$ , by previous proposition we have  $S_fx^{-1} = S_f$  and right multiplying by  $x$  we get  $S_f = S_fx$  with which we prove that also  $x^{-1}S_fx = S_f$  i.e. that also  $x^{-1} \in C_f$ . Let  $x, y \in C_f$  then  $xyS_fy^{-1}x^{-1} = S_f$  thus also  $xy \in C_f$ .  $\square$

We remark that in general if  $x$  leaves invariant spinor space  $S_f$  nothing can be said on its properties on a different  $S_{f'}$ , moreover this subgroup is not normal as it is simple to see.

In turn  $C_g^*$  contains another subgroup of transformations that stabilizes *any* spinor space  $S_f$ ; in this case one can prove that its generic element has the form

$$x = \sum_a \alpha_a \Psi_{aa} \quad \alpha_a \in \mathbb{F} - \{0\}$$

where  $\Psi_{aa}$  are the primitive idempotents; these elements form an abelian subalgebra and a normal subgroup of  $C_g^*$ . In a certain sense this group is the equivalent of the Clifford Lipschitz group in the sense that whereas the former is a subgroup of  $C_g^*$  that stabilizes vectors the latter stabilizes spinor spaces. The study of these groups will be pursued in a forthcoming paper.

## 5. Conclusions

We have proved that all orthogonal transformations of an even dimensional vector space  $V$  can be seen as the restrictions of inner automorphisms of  $\mathcal{Cl}(g)$ . We are thus allowed to assume that also a spinor  $\varphi$  must transform as  $x\varphi x^{-1}$  and that, in some cases like e.g.  $P$  and  $T$ , this has the effect of moving spinor  $\varphi \in S_f$  to  $x\varphi x^{-1} \in S_{-f}$ . This has no influence on the solutions of equations like  $v\varphi = 0$  but the moved spinor  $x\varphi x^{-1}$  may have opposite chirality.

The perspectives appear interesting but many things remain to be done to complete this study, one for all the classification of continuous transformations of  $V$  since it is a simple exercise to verify that whereas all automorphisms generated by an odd number of generators, like e.g.  $\varphi \rightarrow e_i \varphi e_i^{-1}$ , move the spinor space of  $\varphi$ , automorphisms where the generators appear in couples, like e.g.  $\varphi \rightarrow (e_{2i-1} e_{2i}) \varphi (e_{2i-1} e_{2i})^{-1}$ , do not move the spinor space of  $\varphi$ . This and other issues on the group of transformations that stabilize spinors will be tackled in ongoing work.



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